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Eigenanalysis for Membranes with Stringers Using the Methods of Fundamental Solutions and Domain Decomposition

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Abstract: We use a meshless numerical method to analyze the eigenanalysis of thin circular membranes with degenerate boundary conditions, composed by different orientations and structures of stringers. The membrane eigenproblem is studied by solving the two-dimensional Helmholtz equation utilizing the method of fundamental solutions and domain decomposition technique as well. The method of singular value decomposition is adopted to obtain eigenvalues and eigenvectors of the resulting system of global linear equation. The proposed novel numerical scheme was first validated by three circular membranes which are structured with a single edge stringer, two opposite edge stringers and an internal stringer. Present results for those three cases match very well with the solutions obtained by the analytical approach as well as by methods of dual boundary element, and finite element. The analysis is then extended to solve a completely new problem of a circular membrane with a cross stringer at the center of the membrane. We illustrate the proposed innovative numerical scheme which is simpler and more efficient to solve Helmholtz problems with degenerate boundary conditions. The good features of this scheme are not depending upon the treatments of mesh, singularity, hypersingularity, numerical integration and iterative procedure, which are generally required by other conventional mesh-dependent methods.

keyword: eigenanalysis, Helmholtz equation, method of fundamental solutions, domain decomposition method, degenerate boundary condition

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1 Introduction

In recent years, there has been an increasing interest in the research issues of meshless numerical methods for solving partial differential equations. In general, the meshless methods can be divided loosely into two categories: the boundary-type and the domain-type. One commonly used scheme in the domain-type category is the multiquadrics method [Kansa (1990a, 1990b)] which is chosen from the family of the radial basis functions (RBFs). On the other hand, a popularly utilized method in the boundary-type category is the meshless local Petrov-Galerkin (MLPG) method [Atluri (2004)]. In this paper, we applied the method of fundamental solutions (MFS), which is another boundary-type meshless numerical method, to solve Helmholtz problem. The MFS has been applied to solve the potential flow problems [Johnston and Fairweather (1984), Karageorghis (1989), Golberg and Chen (1998)], the Helmholtz equations [Kondapalli *et al.*(1992), Golberg and Chen (1998), Karageorghis (2001), Young and Ruan (2005)], the biharmonic equations [Karageorghis and Fairweather (1987, 1988, 1989)], the Poisson equations [Golberg (1995)], the diffusion equations [(Young *et al.* (2004a, 2004b)], and the Stokes flow problems [Tsai *et al.* (2002), Alves and Silvestre (2004),Young *et al.* (2004c)]. The MFS has been widely used to solve the Helmholtz problems without degenerating scales with simple boundary conditions [Young and Ruan (2005); Young *et al*. (2005a)], in which some original ideas of the MFS for Helmholtz problems are stemmed from the boundary element method (BEM). Recent researches on the BEM for Helmholtz problems can be found from: Callsen *et al.* (2004), Qian *et al.* (2004a, 2004b), and Yan *et al.* (2005). However, the applications of the MFS for Helmholtz equations with degenerate boundary conditions have never been investigated. Hence in this paper we will emphasize the development of the MFS to solve Helmholtz problems with degenerate boundary conditions, such as a circular membrane with stringers inside in one case, or seepage flows with sheet piles in another application [Young, *et al.* (2005b)]. Furthermore, the singularity near the degenerate boundary will be circumvented by using the domain decomposition method (DDM).

The numerical treatment of degenerate boundary value problems is always a challenging task in the field of computational engineering. Especially, it is rather difficult to use directly the conventional mesh-dependent methods such as the finite element method (FEM), the finite difference method (FDM), the finite volume method (FVM) and the BEM to deal with the degenerate boundary problems. Some sophisticated remedies and modifications must be taken to overcome the difficulty of the degenerate boundary. For examples, Jones *et al.* (2001) proposed a hybrid approach to treat degenerate boundary conditions associated with the seepage flownet at sheet piles. Rasmussen and Yu (2003) utilized a complex variable BEM to solve potential problems with degenerate boundary conditions, and they observed the failures of a one-to-one correspondence for the multiple capture zones. Moreover, Chen *et al.* (2001) solved the degenerate scale problems using the BEM with separable kernels, and Cruse (1987) solved the crack degeneracy problems by employing the hypersingular BEM.

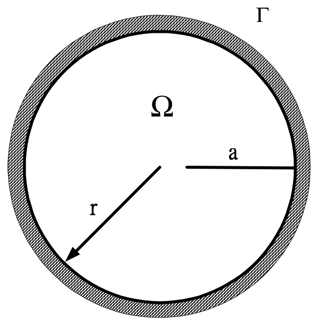
It is a rather difficult task to deal with the acoustical eigenanalysis in a computational domain with degenerate boundary conditions. Givoli and Vigderguaz (1994) applied the FEM to analyze the Helmholtz problem with degenerate boundary conditions by mesh refinement near the degenerate boundary. Moreover, Kirkup (1997) studied a two-dimensional boundary value problem with a discontinuity in the domain by recasting the governing partial differential equation into an integral equation. On the other hand, Chen *et al.* (1999) utilized the dual boundary element method (DBEM) by making use of hypersingular integral equations to solve the degenerate boundary generated in the presence of a stringer in a circular membrane. In a similar work, Chen *et al.* (2003) obtained eigensolutions for circular membranes with stringers by employing the conventional BEM along with the technique of the singular value decomposition (SVD).

In the present work, we concentrate on developing a meshless numerical method to handle the Helmholtz problems with degenerate boundary conditions using the DDM [Chan *et al.* (1989, 1990)]. We decompose the computational domain into a number of sub-domains to avoid the singularity near the degenerate boundary. After DDM is performed, the MFS is then used to obtain the system matrices for all sub-domains. By imposing the connection conditions and the boundary conditions for the stringers, a global matrix system of linear algebraic equations for the whole computational domain is obtained. Afterwards the SVD is then applied to seek the eigenvalues and eigenvectors. Herein the DDM has avoided the singularity problem, at the same time it also guarantees smooth solutions in the entire domain even without iterative procedures, as always required by other conventional mesh-dependent methods. Hence the present method is proved to be simple and computationally efficient to solve the degenerate boundary without handling the problems of singularities, meshes, and numerical integrations. We have successfully undertaken the potential problems with the degenerate boundary conditions to solve the seepage flownet analysis by using the MFS and DDM [Young *et al.* (2005b)]. The details about the proposed method are discussed in Sec. 3 after outlining the governing equations in Sec. 2. The results delineated in Sec. 4 are compared with analytical solutions and the results using the FEM [Givoli and Vigdergauz (1994)], DBEM [Chen *et al.* (1999)] and conventional BEM [Chen *et al.* (2003)]. Finally, concluding remarks are summarized in Sec. 5.

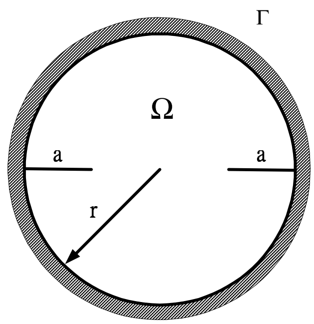
2 Governing equations

In order to test the numerical scheme, we consider problems of a thin vibrating membrane with different orientation of stringers (Fig.1). The presence of stringers in the membrane gives rise to the degenerate boundaries. The flat elastic membrane is assumed to occupy a finite domain Ω. The membrane contains some straight fixed-edge stringers, which may either reach the boundary or remain in the interior of the domain. A prescribed lateral displacement of the boundary results in small-amplitude time harmonic lateral vibration of the membrane with wave number given by $k = \omega/\sqrt{T/\rho}$, where T is the tension in the membrane, ρ is its mass density, and ω is the frequency of motion. On the stringers, which are denoted by Γ_d, the lateral displacement ϕ is vanished. The above membrane analysis is governed by the two-dimensional homogeneous Helmholtz equation given as

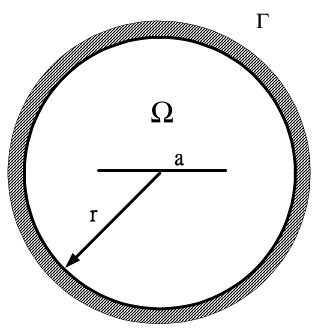
$$\nabla^2\phi(\vec{\mathbf{x}}) + k^2\phi(\vec{\mathbf{x}}) = 0, \quad \text{in} \quad \Omega \tag{1}$$



(a)



(b)



(c)

Figure 1 : A vibrating membrane (a) with a single stringer (b) with two opposite edge stringers (c) with an internal stringer

where Ω is the domain of interest, $\vec{\mathbf{x}}$ represents the location of the field points, ϕ is the displacement and k is the wave number. The boundary conditions are given as follows:

$$\phi(\vec{\mathbf{x}}) = \overline{\phi}, \quad \text{on} \quad \Gamma_D \tag{2}$$

$$\frac{\partial \phi(\vec{\mathbf{x}})}{\partial n_x} = \bar{t}, \quad \text{on} \quad \Gamma_N \tag{3}$$

where Γ_D is the essential (Dirichlet) boundary with specified displacement, Γ_N is the natural (Neumann) boundary where the outward normal derivative of the displacement in the n_x outward normal direction is specified, and Γ_D and Γ_N comprise the whole boundary Γ of the domain Ω. In this paper the boundary conditions employed are given by Eq. 2 that $\overline{\phi}$ is equal to zero for both solid walls and stringers. That is, the only condition used in this study is the Dirichlet boundary condition. The Neumann boundary condition is used for the connection of the DDM.

3 Numerical method

The MFS makes use of boundary values to obtain the solution in the interior of the computational domain. When a membrane is considered as a single computational domain, it is required to use the hypersingularity in the case of degenerate boundary conditions. It is necessary to satisfy the degenerate boundary conditions on the stringers. In the proposed numerical procedure, the hypersingular problem is avoided by using the principles of the DDM. The membrane is decomposed into a number of sub-domains in which the boundary conditions for the stringers also appear explicitly. Since the MFS uses the boundary values only for the interior domain solution, the individual sub-domains are solved easily with known boundary conditions. In the present case, the free-space Green's function for the Helmholtz equation is written as follows:

$$\nabla^2 G(\vec{\mathbf{x}}) + k^2 G(\vec{\mathbf{x}}) = -\delta(\vec{\mathbf{x}} - \vec{\xi}) \tag{4}$$

where $G(r_{ij}) = \frac{-i}{4} H_0^{(2)}(kr_{ij}) = \frac{-i}{4}\left[J_0(kr_{ij}) - iY_0(kr_{ij})\right]$ is the fundamental solution, $\delta(\vec{\mathbf{x}} - \vec{\xi})$ is the Dirac delta function, $\vec{\mathbf{x}}$ is the position of the field point, and $\vec{\xi}$ is the position of the source point. In the fundamental solution, $H_0^{(2)}$ is the Hankel function of the second kind of order zero, J_0 is the Bessel function of the first kind of order zero, and Y_0 is the Bessel function of the second kind of order zero. Using the above expression, an approximate solution for the Helmholtz equation can be obtained as

$$\phi(x_i, y_i) = \sum_{j=1}^{N} \alpha_j^{\phi} G(r_{ij}) \tag{5}$$

and the normal derivative is given by the following ex-

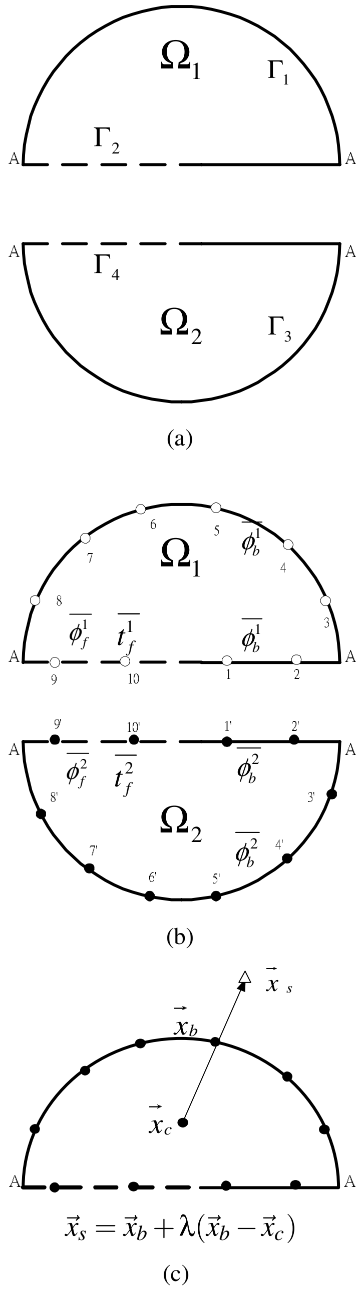


Figure 2 : Figure sketch of the domain decomposition

pression:

$$\frac{\partial \phi(x_i, y_i)}{\partial n_i} = \sum_{j=1}^{N} \alpha_j^{\phi} \frac{\partial G(r_{ij})}{\partial n_i}$$

$$\frac{\partial G(r_{ij})}{\partial n_i} = \frac{\partial G(r_{ij})}{\partial r} \frac{\partial r}{\partial n_i} = -\left[\frac{\vec{r}\cdot\vec{n}}{4r}k\right]\left[Y_1(kr_{ij}) + iJ_1(kr_{ij})\right] \tag{6}$$

where J_1 is the Bessel function of the first kind of order one, Y_1 is the Bessel function of the second kind of order one, $\phi(x_i, y_i)$ is the displacement of the i-th node, r_{ij} is the distance between the i-th and the j-th nodes and is defined as $r_{ij} = |\vec{r}_i - \vec{r}_j|$, α_j^{ϕ} is the undetermined coefficients and N is the number of source points. The distributions of the source nodes can be arranged by the following equation;

$$\vec{x}_s = \vec{x}_b + \lambda(\vec{x}_b - \vec{x}_c) \tag{7}$$

where $\vec{x}_b$ and $\vec{x}_s$ are the spatial coordinates of the boundary and source nodes, respectively. $\vec{x}_c$ is the center position of the computational domain and λ is a pre-assigned parameter. Once the parameter, λ, is determined, the distributions of the source nodes is obtained as depicted in Fig. 2(c).

The application of the MFS along with the DDM to solve the degenerate problem can be explained by considering a membrane with a stringer shown in Fig. 1(a). The stringer generates a degenerate boundary where the displacements are zero. By the proposed numerical scheme, the membrane is assumed to consist of two sub-domains Ω_1 and Ω_2 divided along the section A-A across the stringer as shown in Fig. 2(a). To resolve the degenerate boundary, the computational domain is thereby decomposed into two sub-domains. However, the computational domain encompassing the thin membrane is a single spatial identity. Hence, while applying the MFS to find the solution for the Helmholtz equation, a care must be taken to ensure the continuity of the variables in the sub-domains. In order to satisfy this connecting condition, the displacements, ϕ, and the flux values, t, on the common boundaries of the sub-domains are assumed to be the same. Owing to the above two relationships, the sub-domains Ω_1 and Ω_2 can be related such that they constitute a single computational domain. For further details of the principles of the DDM Chan *et al.* (1989, 1990) are referred. The individual boundary of the separation domain is shown in Fig.2 (b). $\overline{\phi_b^1}$, $\overline{\phi_f^1}$ and $\overline{t_f^1}$ belong to Ω_1. Similarly, $\overline{\phi_b^2}$, $\overline{\phi_f^2}$ and $\overline{t_f^2}$ belong to Ω_2. Let points 1…8 be boundary points in $\overline{\phi_b^1}$ and let 9, 10 be fictitious points in $\overline{\phi_f^1}$ and $\overline{t_f^1}$. Let points 1'…8' be boundary points in $\overline{\phi_b^2}$ and let 9', 10' be fictitious points in $\overline{\phi_f^2}$ and $\overline{t_f^2}$. From Eqs. 5 and 6, approximate solutions using the MFS are expressed in matrix forms for the following individual

Table 1 : The first eight eigenvalues for the membrane with a single edge stringer (Nodes = 80)

Method	Eigenvalues (a=1.0)							
	k1	k2	k3	k4	k5	k6	k7	k8
MFS	3.13	3.82	4.49	5.13	5.75	6.29	6.36	6.97
DBEM	3.13	3.83	4.49	5.14	5.75	6.29	6.36	6.96
FEM	3.14	3.82	4.48	5.12	5.74	6.27	6.35	6.95
EXACT	π	3.83	4.50	5.14	5.76	2π	6.38	6.92

sub-domains:

$$\left[A_{ij}^{\Omega_1}\right]\left\{\alpha_j^{\Omega_1}\right\} = \begin{bmatrix} A_{ij}\big|_{\Gamma_1} \\ A_{ij}\big|_{\Gamma_2} \end{bmatrix}\left\{\alpha_j^{\Omega_1}\right\} = \begin{Bmatrix} \overline{\phi_b^1} \\ \overline{\phi_f^1} \\ \overline{t_f^1} \end{Bmatrix} \tag{8}$$

$$\left[A_{ij}^{\Omega_2}\right]\left\{\alpha_j^{\Omega_2}\right\} = \begin{bmatrix} A_{ij}\big|_{\Gamma_3} \\ A_{ij}\big|_{\Gamma_4} \end{bmatrix}\left\{\alpha_j^{\Omega_2}\right\} = \begin{Bmatrix} \overline{\phi_b^2} \\ \overline{\phi_f^2} \\ \overline{t_f^2} \end{Bmatrix} \tag{9}$$

where coefficient matrices $[A_{ij}^{\Omega_1}]$ and $[A_{ij}^{\Omega_2}]$ contain the summation of RBFs for the Helmholtz equation for the sub-domains Ω_1 and Ω_2, respectively. Γ_1 and Γ_2 are the physical and fictitious boundaries on sub-domains Ω_1; Γ_3 and Γ_4 are the physical and fictitious boundaries on sub-domains Ω_2; $\overline{\phi_f}$, $\overline{t_f}$ are the values of the Dirichlet and Neumann conditions on the fictitious boundary; and $\{\alpha_j^{\Omega_1}\}$ and $\{\alpha_j^{\Omega_2}\}$ are the intensities of sources of Ω_1 and Ω_2. We must solve Eqs. (8) and (9) by satisfying the following connection conditions at the interfaces of the sub-domains:

$$\left\{\overline{\phi_f^1}\right\} = \left\{\overline{\phi_f^2}\right\} \tag{10}$$

and

$$\left\{\overline{t_f^1}\right\} = \left\{\overline{t_f^2}\right\} \tag{11}$$

By using the DDM and combining from Eq. 8 to Eq.11, the final solution matrix for the entire computational domain can be obtained as:

$$\begin{bmatrix} A_{ij}\,|_{\Gamma_1} & 0 \\ 0 & A_{ij}\,|_{\Gamma_3} \\ A_{ij}\,|_{\Gamma_2} & -A_{ij}\,|_{\Gamma_4} \end{bmatrix} \begin{bmatrix} \alpha_j^{\Omega_1} \\ \alpha_j^{\Omega_2} \end{bmatrix} = \begin{bmatrix} \phi_{\overline{\phi_b^1}} \\ \phi_{\overline{\phi_b^2}} \\ 0 \end{bmatrix} \tag{12}$$

Thus, Eq(12) is a system of global linear equations for the Helmholtz problem in the whole computational domain. In the case of a membrane vibration, we need to seek eigenvalues for which the intensities $\{\alpha_j^{\Omega_1}\}$ and $\{\alpha_j^{\Omega_2}\}$are nontrivial. Using the SVD technique [Teukolsky *et al.* (1992)] for the system matrix in Eq. 12, we can plot the minimum singular values versus the wave number k. From the plot, the eigenvalues are thus obtained [Chen *et al.* (2003)]; and the corresponding eigenvectors, $\{\alpha_j^{\Omega_1}\}$ and $\{\alpha_j^{\Omega_2}\}$ are also solved by the SVD algorithm. Then the eigenmodes is obtained by utilizing Eq. 5. It is worth to notice that iterative procedures and treatment of hypersingularity are not needed anymore when the DDM and MFS are employed in this degeneracy problem.

4 Results and discussions

We now test the proposed innovative numerical method for validation of circular membranes with stringers at different orientations and structures. Various examples discussed by Givoli and Vigdergauz (1994) and Chen *et al.* (2003) are considered for validation purposes as shown in Fig. 1(a)-(c). Depending upon the locations of these stringers, the membrane displacements will change accordingly during vibration. We consider a circular membrane with radius $r = 1$ in the analysis.

4.1 Validations for membranes with (a) single, (b) two opposite edge and (c) internal stringers

After Eq. (12) is solved by using the SVD, the wave number is plotted against the minimum singular values of the system matrix for a circular membrane with a single stringer as shown in Fig. 3. The eigenvalues are obtained as the minimum singular values of the above graph. Figs. 4 and 5 show similar plots for cases of a circular membrane with two opposite edge stringers and with an internal stringer, respectively. To check the accuracy of the proposed method, the obtained eigenvalues are compared with analytical and numerical solutions as shown in Tab. 1. The proposed numerical scheme predicts the first eight eigenvalues of the circular membrane precisely with a single stringer. It is also very comparable with the results of the FEM [Givoli and Vigderguaz

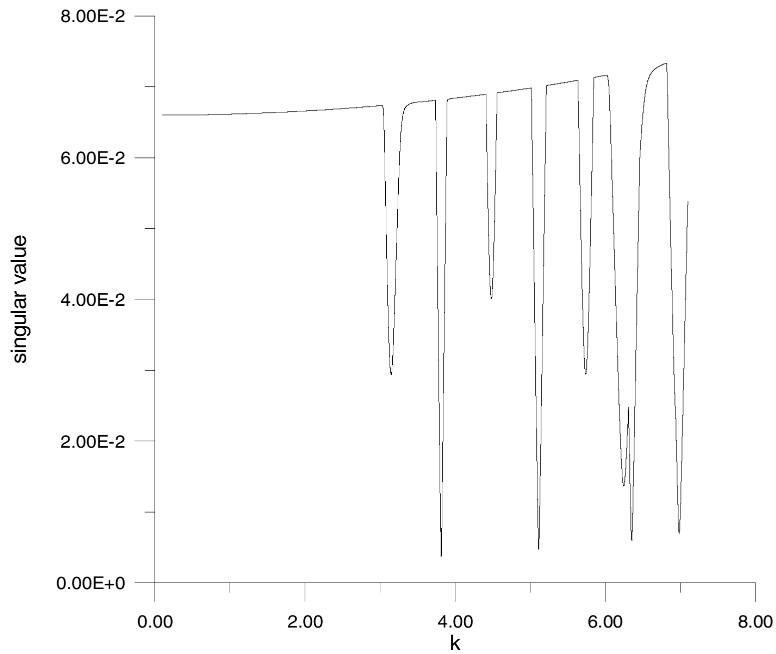


Figure 3 : The singular value versus k (wave number) for the membrane with a single stringer (Nodes = 80)

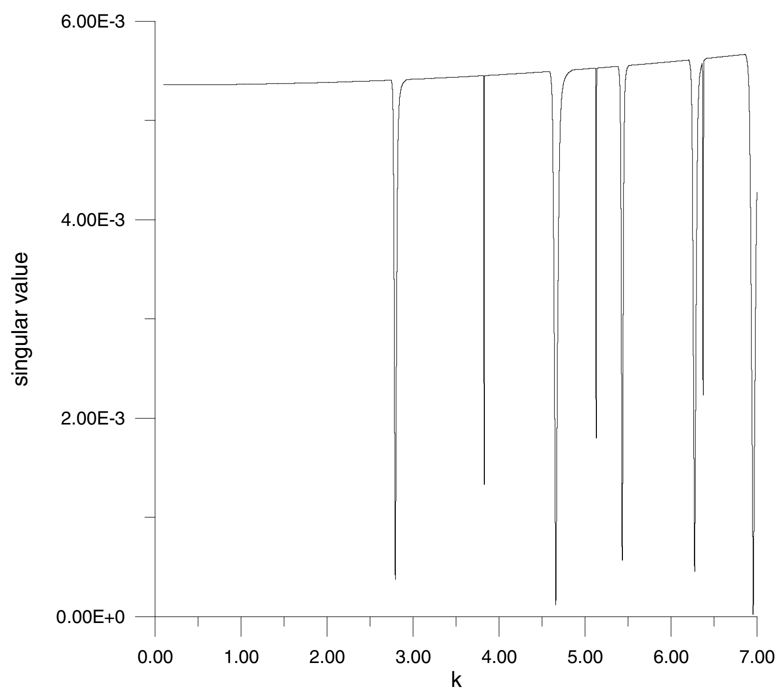


Figure 4 : The singular value versus k (wave number) for the membrane with two opposite edge stringers (Nodes = 80)

Table 2 : The first four critical wave numbers for the membrane with a single edge stringer (squares), k^2 (Nodes = 80)

		a=0.1	a=0.2	a=0.3	a=0.4	a=0.6	a=0.7	a=0.8	a=0.9
Mode	Method	k*k	k*k	k*k	k*k	k*k	k*k	k*k	k*k
mode1	MFS	5.919	6.31	7.054	8.1	11.76	13.84	14.52	14.68
	DBEM	5.91	6.3	7.02	8.15	11.99	13.92	14.65	14.71
mode2	MFS	14.65	14.63	14.6	14.54	14.66	14.67	25.41	26.32
	DBEM	14.72	14.72	14.72	14.72	14.71	14.71	25.68	26.41
mode3	MFS	15.26	16.7	18.54	19.77	20.38	22.7	26.37	40.76
	DBEM	15.22	16.69	18.57	19.94	20.5	22.65	26.41	40.73
mode4	MFS	26.32	26.28	26.14	26.13	26.42	26.42	37.88	49.14
	DBEM	26.45	26.44	26.44	26.43	26.43	26.42	38.28	49.32

Table 3 : The first eight eigenvalues for the membrane with two opposite edge stringers (Nodes = 80)

Method	Eigenvalues (a=0.5)							
	k1	k2	k3	k4	k5	k6	k7	k8
MFS	2.79	3.83	4.66	5.13	5.43	6.28	6.37	6.96
DBEM	2.79	3.83	4.66	5.14	5.44	6.28	6.38	6.97
FEM	2.82	3.83	4.67	5.12	5.43	6.26	6.35	6.96

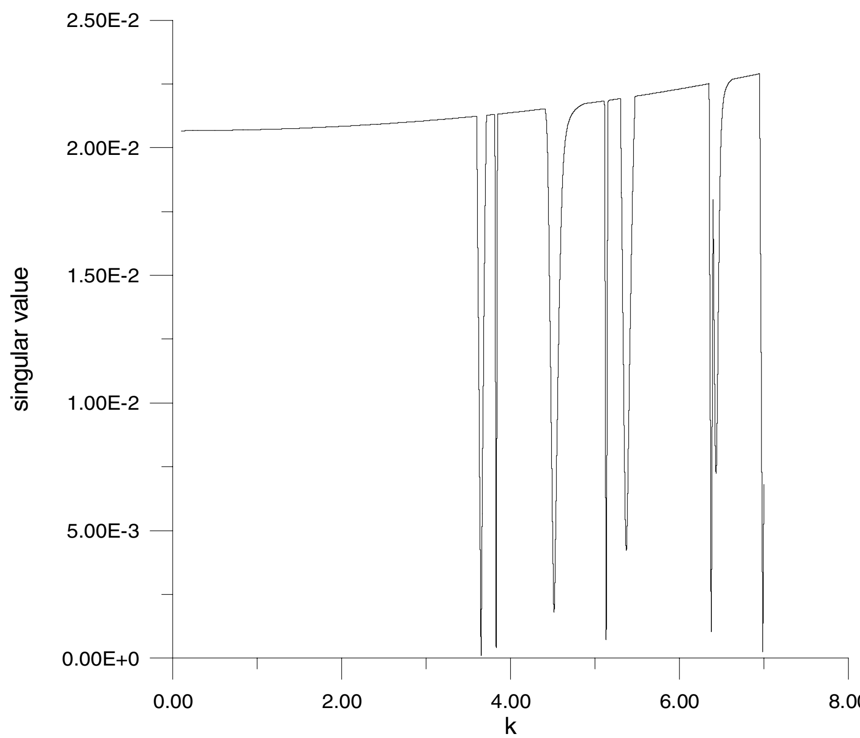


Figure 5 : The singular value versus k (wave number) for the membrane with an internal stringer (Nodes = 80)

(1994)], DBEM [Chen *et al.* (1999)] and exact solutions. It is also predicted that the values of the first four critical wave numbers for the circular membrane with a single stringer as a function of radius of the membrane, as given in Tab. 2. The comparison of the proposed results with those of FEM and DBEM solutions reveals that the proposed method is able to predict the eigenvalues accurately. The above analysis can be extended to the case of a circular membrane with two opposite edge stringers. The first eight eigenvalues for $a = 0.5$ and variation of

Table 4 : The first three critical wave numbers for the membrane with two opposite edge stringers (squares), k^2 (Nodes = 80)

		a=0.05	a=0.1	a=0.15	a=0.2	a=0.3	a=0.35	a=0.4	a=0.45
Mode	Method	k*k	k*k	k*k	k*k	k*k	k*k	k*k	k*k
mode1	MFS	5.83	6.01	6.4	6.92	9.24	10.75	12.67	14.15
	DBEM	5.85	6.05	6.41	6.99	9.05	10.73	12.66	14.17
mode2	MFS	14.68	14.69	14.68	14.67	14.67	14.67	14.67	14.67
	DBEM	14.74	14.72	14.72	14.72	14.71	14.71	14.71	14.71
mode3	MFS	14.92	15.77	17.31	19.28	24.3	25.6	26.21	26.34
	DBEM	14.95	15.87	17.36	19.42	24.07	25.61	26.34	26.42

Table 5 : The first eight eigenvalues for the membrane with an internal stringer (Nodes = 80)

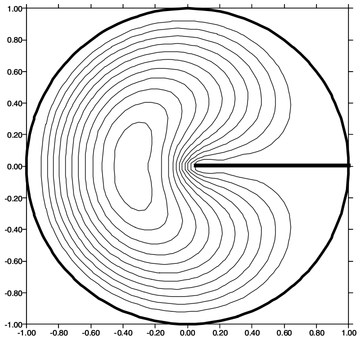
Method	Eigenvalues (a=0.8)							
	k1	k2	k3	k4	k5	k6	k7	k8
MFS	3.65	3.83	4.52	5.14	5.37	6.38	6.43	6.99
DBEM	3.63	3.84	4.46	5.14	5.33	6.39	6.42	7
FEM	3.66	3.81	4.55	5.07	5.38	6.27	6.35	6.86

wave number with length of 'a' are compared with the results of FEM and DBEM as described in Tab. 3 and 4, respectively. The values of eigenvalues determined by the proposed method are also in close agreement with the results of FEM and DBEM. In Tab. 5, we address results for the eigenvalues obtained for a circular membrane with an internal stringer. We have good agreement between the FEM and DBEM solutions. The number of field points, $N = 80$ and the parameter $\lambda = 0.02$ are chosen in all above three validation cases.

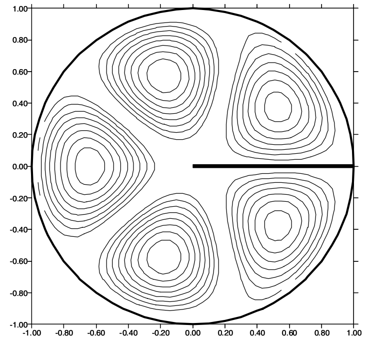
After obtaining the eigenvalues, we consider the eigenmodes for the problems. The eigenmodes of the membrane is affected by the wave number as well as the presence of stringers. The displacement contours, also called the mode shapes, for a circular membrane with a single stringer are depicted in Fig. 6 for various wave numbers. The value of the wave number, k increases the mode shape of the membrane as seen in the above figure. The presence of the stringer inside the membrane alters the mode shapes for different values of wave numbers. The above mode shapes match in good qualitative comparison with the results of Givoli and Vigdergauz (1994) and Chen *et al*. (2003). The proposed numerical scheme predicts the displacements very accurately in the presence of a degenerate boundary generated by a stringer. We sketch the mode shapes for the membrane with two opposite edge stringers with $a = 0.5$ in Fig. 7 for different values of wave number. A comparison between Figs. 6 and 7 shows that the two opposite edge stringers modify the displacement shapes as expected. Though the mode shapes for the wave number equal to 3.82 in Fig. 6 and for the wave number equal to 3.83 in Fig. 7 show similar shapes, the increase in the stiffness of the membrane with two opposite edge stringers has been correctly predicted by the form of wide spaced contours in the later figure. These mode shapes agree qualitatively with the results of Givoli and Vigdergauz (1994) and Chen *et al.* (2003). Fig. 8 shows the mode shapes obtained by the present numerical scheme for the case of a membrane with an internal stringer with $a = 0.8$ for different values of the wave number. The mode shapes vary with the values of the wave number and also in good qualitative agreement with the existing literature. In all the above figures, the effect of the wave number on the mode shapes is made more significantly only for higher values of wave numbers. The comparison of the proposed results for wave numbers obtained for different orientations of the stringers in the membrane with those of the FEM and DBEM indicate that the proposed numerical scheme is simple, accurate and efficient.

4.2 *Application to a membrane with a cross stringer*

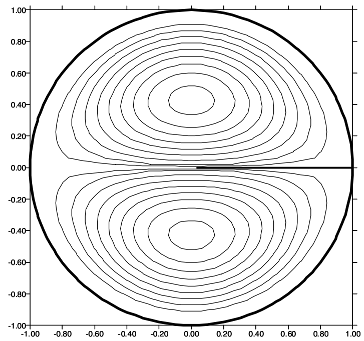
After validating the numerical scheme, an entirely brand new problem is solved to demonstrate the capability of the present scheme. A circular membrane with a cross stringer at the center as shown in Fig. 9 is solved for the eigenmodes with $a = 0.8$. We choose the number of field points, $N = 60$ and the parameter $\lambda = 0.02$ in this case.



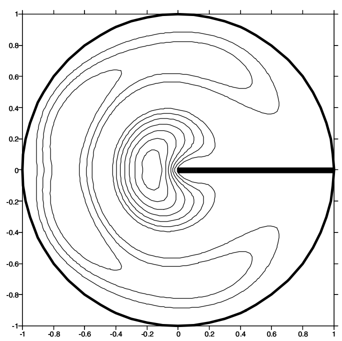
(a) $k = 3.13$



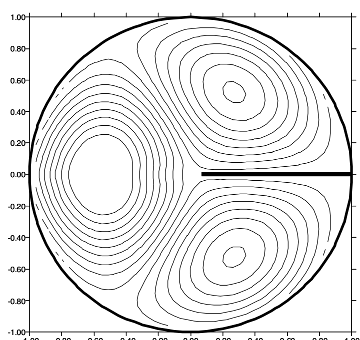
(b) $k = 5.75$



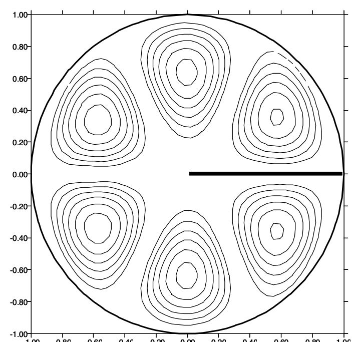
(c) $k = 3.82$



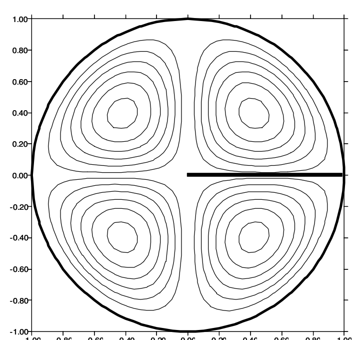
(d) $k = 6.29$



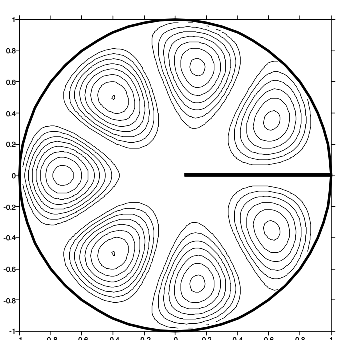
(e) $k = 4.49$



(f) $k = 6.36$

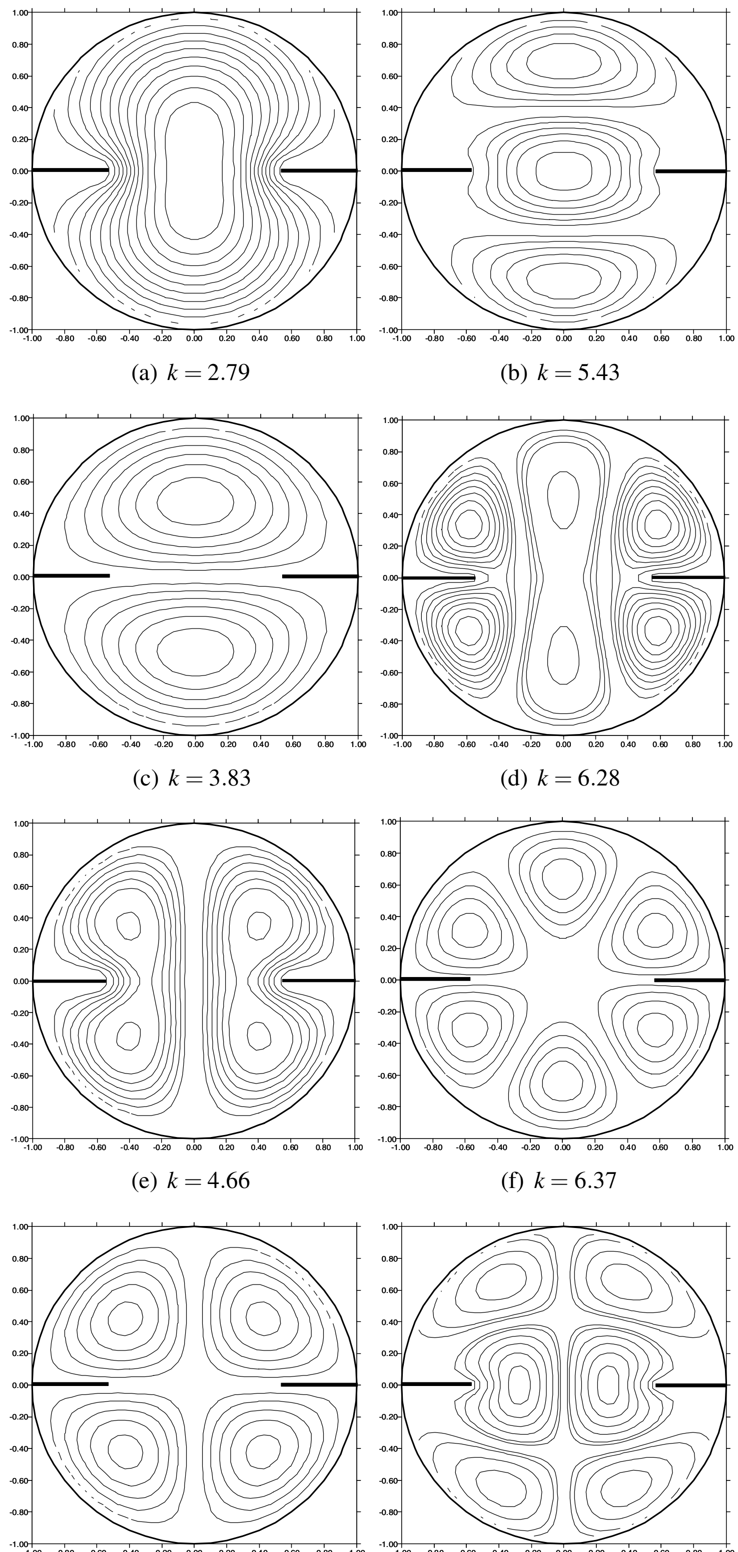


(g) $k = 5.13$



(h) $k = 6.97$

Figure 6 : The first eight modes of the single stringer with $a = 1$ by MFS (Nodes = 80)



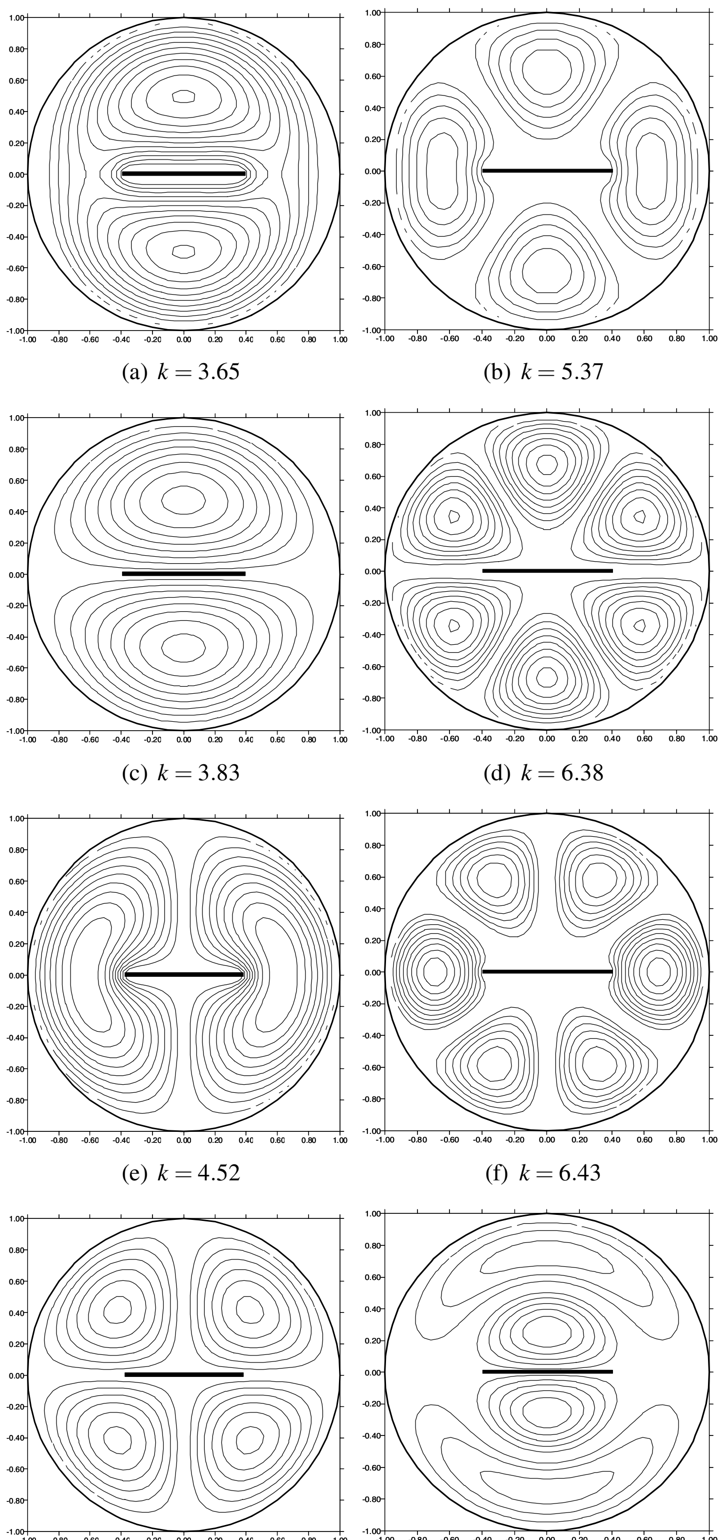
(a) $k = 2.79$ (b) $k = 5.43$

(c) $k = 3.83$ (d) $k = 6.28$

(e) $k = 4.66$ (f) $k = 6.37$

(g) $k = 5.13$ (h) $k = 6.96$

Figure 7 : The first eight modes of the two-edge stringer with $a = 0.5$ by MFS (Nodes = 80)



(a) $k = 3.65$ (b) $k = 5.37$

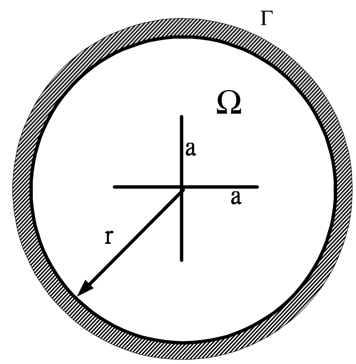
(c) $k = 3.83$ (d) $k = 6.38$

(e) $k = 4.52$ (f) $k = 6.43$

(g) $k = 5.14$ (h) $k = 6.99$

Figure 8 : The first eight modes of the center stringer with $a = 0.8$ by MFS (Nodes = 80)

Figure 9 : A vibrating membrane with a cross stringer.



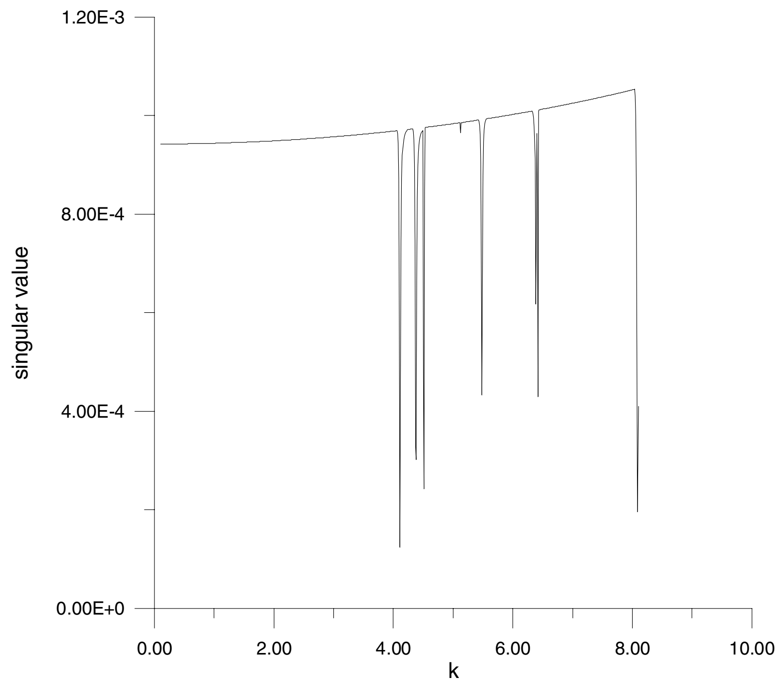
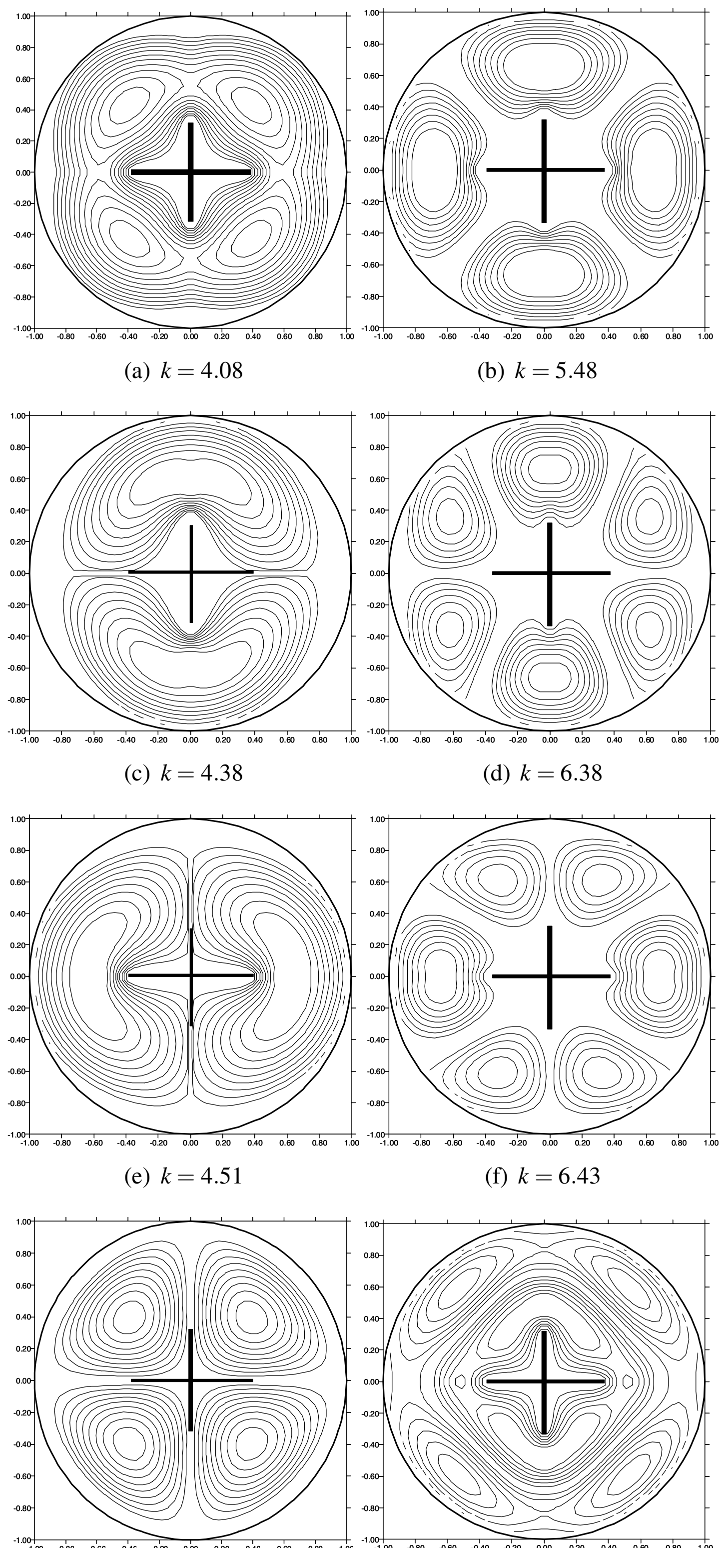


Figure 10 : The singular value versus k (wave number) for the membrane with a cross stringer (Nodes=60)

The eigenvalues of this problem as depicted in Fig. 10 are also obtained by using the SVD technique. After determining the eigenvalues, the eigenmodes are also obtained as before and illustrated in Fig. 11. In all other cases discussed in the previous section, either a single stringer is present or two stringers are present separately without giving any reinforcement. However the presence of a cross stringer at the center of the membrane has increased the stiffness of the membrane. Therefore the eigenmodes of displacement patterns vary widely with the eigenvalues comparing with the previous cases.

4.3 Comparison of wave number vs. mode for different stringer positions

When a stringer is placed on the circular membrane, the eigenvalues vary with respect to the frequency modes. The first four modes highly influence the displacement patterns of the membrane. It is important to understand that how the orientation of a stringer will change the displacement contours so that the stringer position can be decided depending upon the functional requirement of the membrane. Without any stringer placed on the membrane, the displacement behavior of the membrane is expected to be symmetric. Fig. 12 describes the variation of the wave number with respect to mode number for a



(a) $k = 4.08$ (b) $k = 5.48$

(c) $k = 4.38$ (d) $k = 6.38$

(e) $k = 4.51$ (f) $k = 6.43$

(g) $k = 5.12$ (h) $k = 8.08$

Figure 11 : The first eight modes of the cross stringer with $a = 0.8$ by MFS (Nodes = 60)

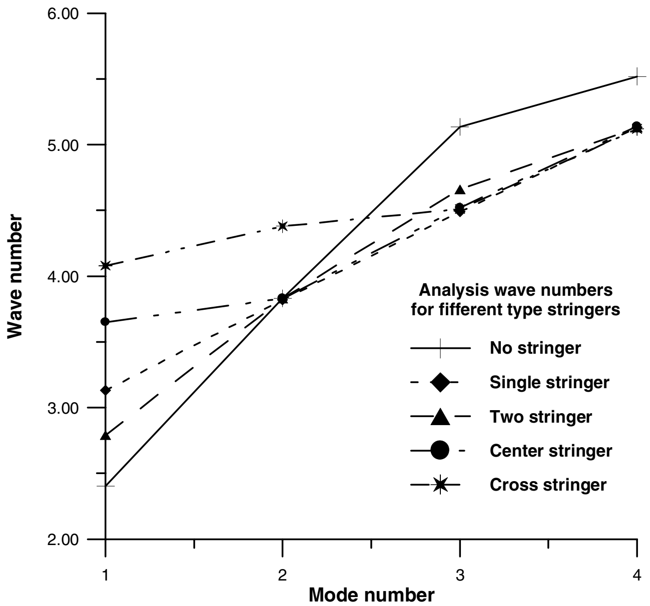


Figure 12 : Comparison of wave number for different position of stringers

circular membrane without stringers and with stringers placed at different orientations. For the comparison purposes, the length of the stringers is assumed to be constant for all cases. The wave number for the first mode is the minimum for membrane without stringers, whereas it is the maximum for the case of the cross stringer. The wave number varies linearly with mode number for all the cases up to the second mode, where the wave number of all the cases, except for the cross stringer, all converge to the same value. However the value of the cross stringer is slightly increased. After the third mode, the wave numbers for all the cases converge to the same value but being lower than the case for a membrane without stringers. We observe from Fig.12 that the presence of stringers affects to change the value of the wave number only at the initial modes.

5 Conclusions

The two-dimensional homogeneous Helmholtz equation, governing the eigenmode behaviors of a thin circular membrane with different orientations and structures of stringers, has been solved using the MFS together with the DDM. This combination has enabled us to deal with the degenerate boundary conditions generated as a result of the presence of stringers in the membrane. The eigenvalues and eigenmodes are computed using the SVD for circular membranes with a single stringer, two opposite edge stringers, and an internal stringer. The first eight eigenmodes for the circular membrane with a single stringer match very well with other numerical and analytical results available in the literature. The eigenvalues and eigenmodes for the membranes with two opposite edge stringers and an internal stringer are also in close agreements with other numerical results.

After validating the numerical scheme for the existing examples, the eigenvalues and the eigenmodes are predicted for the membrane with a cross stringer. The mode shapes obtained for the new case show different displacement contours since the membrane center is reinforced by the formation of a cross stringer. Therefore the displacement patterns are different from the three validation cases as expected.

A comparative study to understand the effect of the presence of stringers on the membrane indicates that the stringer affects the eigenvalues only during the initial modes. This study demonstrates that the present novel numerical scheme is capable to deal with degenerate Helmholtz boundary problems without needs to consider the meshes, singularities, hypersingularity, numerical integrations, and iterative procedures, which are generally required by the conventional mesh-dependent numerical methods such as BEM and FEM.

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